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1993 J. Phys.: Condens. Matter 5 707

(http://iopscience.iop.org/0953-8984/5/6/007)

J. Phys.: Condens. Matter 5 (1993) 707–716. Printed in the UK

Quantum interference from superconducting islands in a mesoscopic solid

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Received 13 August 1992, in final form 16 November 1992

Abstract. By computing Andreev scattering coefficients for a three-dimensional disordered solid containing superconducting inclusions and evaluating a generalized Landauer–Büttiker formula for the two-probe electrical conductance G, an intuitive result is obtained for the dependence of G on the phase of the superconductors. For the simplest case of two inclusions, it is shown that G varies periodically with the phase difference ϕ of the superconducting islands, with period 2π. For more than two inclusions, beating can occur. If the superconductors are decoupled, G varies periodically with time, at the Josephson frequency. If the superconductors are weakly coupled, this behaviour is preceded by a time-independent regime in which ϕ increases with the externally applied voltage, while G changes non-monotonically. It is demonstrated that at least in the presence of single-channel external leads, the ensemble averaged conductance of a highly disordered system varies periodically with period 2π, in contrast with a periodicity of π found for weakly localized systems.

1. Introduction

When a chemical potential difference $\delta\mu$ exists between two superconductors, the phase difference $\phi = \phi_1 - \phi_2$ changes with time according to the Josephson relation $\mathrm{d}\phi/\mathrm{d}t = 2\ \delta\mu/\hbar$. If the Josephson coupling and associated critical current I_c are non-zero, then a range of well known effects associated with superconducting weak links can occur, the precise nature of which depends not only on the geometry of the junction, but also on whether or not the system behaves mesoscopically [1–3]. All of these effects vanish when I_c is negligibly small and the coupling between the superconductors tends to zero. Therefore in this limit, one might expect no interesting physics to be associated with the phase of the superconductors. One exception to this arises in mesoscopic structures, where Andreev scattering of normal electrons provides an extra mechanism for interference from order parameter phases. For a normal disordered material located between two superconducting boundaries, it has been demonstrated [4, 5], that a weak localization contribution to the electrical conductance G and the ensemble averaged conductance $\langle G\rangle$ are oscillating functions of ϕ, with period 2π and π respectively.

For the weakly disordered system examined in [4, 5], electrical conductance is dominated by quasi-particle diffusion, with negligible current carried by the

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superconducting condensate. The aim of this paper is to examine the opposite limit of negligible quasi-particle diffusion, where electrical conductance is dominated by Andreev scattering at superconducting–normal interfaces. To this end, a mesoscopic system containing an arbitrary number of superconducting islands is examined and an intuitive picture developed, which reveals in a transparent manner how oscillatory behaviour arises. For normal systems, a great deal of insight into transport properties has been achieved using an equivalent approach to Green function methods, based on Landauer–Büttiker formulae [6–8]. In what follows, it is demonstrated that by generalizing the two-probe Büttiker formula [8] to account for Andreev scattering, simple perturbation theory can be used to compute the effect on G of quantum interference from the phase of an arbitrary number of superconducting inclusions. For the simplest case of two inclusions, examples of which are given in figure 1, the oscillatory behaviour of G with period 2π is recovered. However, at least for the case of single-channel external leads, at zero temperature, $\langle G\rangle$ is found to oscillate with period 2π, rather than π.

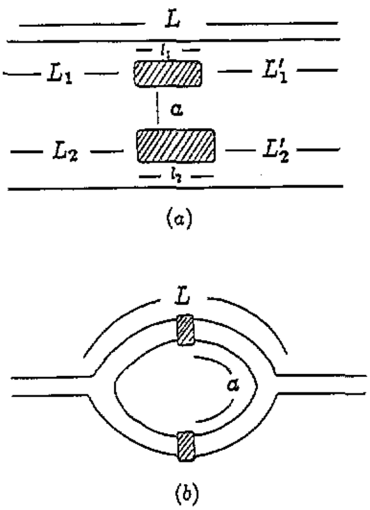

Figure 1. Possible configurations of weakly coupled superconductors (shown shaded) embedded in a normal mesoscopic medium. (*a*) A generic example. (*b*) Two superconductors embedded in the arms of a normal mesoscopic loop.

The origin of such oscillatory effects lies in the fact that by definition, for a mesoscopic structure, the sample size is smaller than, or of the order of, any inelastic scattering length and therefore an excitation maintains phase coherence as it passes through the system. In the presence of a superconducting inclusion with a constant order parameter of phase ϕ_j, as well as normal potential scattering, a particle-like wavepacket can coherently evolve into a hole-like excitation and vice versa. This process, known as Andreev scattering, produces a phase shift of ϕ_j in the outgoing wavepacket. Consequently one expects quite generally that in the presence of more than one inclusion with differing order parameter phases, a variety of new quantum interference effects will arise.

In the simplest case of two inclusions with phases ϕ_1, ϕ_2, in the strong disorder limit, it is shown below that

$$G = 2[A_1^2 + A_2^2 + 2A_1A_2\cos(\phi + \theta_{11})] \tag{1.1}$$

where the amplitudes A_1, A_2 and the phase θ_{11} depend on the size of the inclusions and on details of the quasi-particle diffraction pattern within the sample. This effect has not yet been observed experimentally and before proceeding, it is of interest to ask how it can be distinguished from the AC Josephson effect. Consider a current flowing from right to left, due to a potential difference $\mu_1 - \mu_2$ applied across either of the samples in figure 1. If the Josephson coupling between the islands is negligibly small, then except in the extreme case of a symmetric structure with the current flowing at 90° to a line joining the islands, this will produce a potential difference $\delta\mu$ between the superconductors and therefore ϕ will increase with time at the Josephson frequency $2\ \delta\mu/\hbar$. Consequently the current through the sample will possess an oscillatory component. However, in contrast to the AC Josephson effect, where the oscillating current is bounded by a voltage-independent critical current I_c, the current here is $G(\phi)(\mu_1 - \mu_2)$ and therefore the amplitude as well as the frequency of the oscillatory component increases with applied voltage.

To derive the above result, it is important to realize that the current–voltage relations, which underpin Landauer–Büttiker formulae [6–8] for the electrical conductance of normal mesoscopic structures, must be generalized [9] in the presence of superconductivity. To achieve this generalization [9], two new features were added to the conventional theory [6–8]. First, current–voltage relations were extended to include Andreev scattering [10]. Second, for a single superconducting inclusion, to ensure quasi-particle charge conservation at equilibrium, the chemical potential μ of the superconductor was determined self-consistently. For a collection of superconductors with a spread $\delta\mu$ of potentials centred on μ, the result remains valid, provided $\delta\mu \ll |\mu_1 - \mu_2|$, where μ_1, μ_2 are chemical potentials of external reservoirs supplying the current I flowing through the device.

In [9] generalized four-probe Landauer formulae were obtained at both zero and finite temperature. The simpler two-probe formulae were not written down explicitly, but are implicit in the analysis. Since the latter are more accessible experimentally, the analysis in this paper will focus on the two-probe conductance $G = eI/(\mu_1 - \mu_2)$ and, for simplicity, will be restricted to zero temperature. All reflection and transmission coefficients will be evaluated at energy $E = 0$ (relative to the chemical potential μ), where particle–hole symmetry can be exploited. At this energy, taking into account unitarity of the S-matrix and quasi-particle probability conservation, the only distinct coefficients are [8] R_0, R_a (R_0', R_a') and T_0, T_a (T_0', T_a'), corresponding to normal and Andreev reflection of quasi-particles from the left (right) reservoir and normal and Andreev transmission from the left (right) respectively. These satisfy $R_0 + R_a + T_0 + T_a = R_0' + R_a' + T_0' + T_a' = 1$ and $T_0 + T_a = T_0' + T_a'$ and yield for the two-probe conductance G, in units of $h/2e^2$,

$$G = T_0 + T_a + 2[R_aR_a' - T_aT_a']/[R_a + R_a' + T_a + T_a']. \tag{1.2}$$

In the case of a highly disordered sample, with negligible transmission, this reduces to

$$G = 2/[R_a^{-1} + R_a'^{-1}] \tag{1.3}$$

which is the appropriate form when Andreev scattering short-circuits normal conduction. These expressions are not restricted to one dimension. For $d > 1$, coefficients are obtained by summing over N channels corresponding to different transverse k-vectors in the normal leads [9]. For example if $(R_a)_{i,j}$ is the Andreev reflection coefficient from channel j on the left into channel i on the left, then

$$R_a = \sum_{i,j=1}^{N} (R_a)_{i,j}.$$

The possibility of new dynamical effects arises because, as demonstrated below, reflection and transmission coefficients vary periodically with the phase difference ϕ. To observe this periodicity, the usual criteria for mesoscopic behaviour must apply. At a temperature T, these are $L_{\text{inel}} > L$ and $k_B T t_L/\hbar < 1$, where L_{inel} is the inelastic scattering length and t_L is the average time spent by a quasi-particle within the device, as it passes from one external lead to the other. If D is the diffusion constant, then $t_L = L^2/D = h/E_T$, where E_T is the Thouless energy.

In the linear response regime of the normal system, these are supplemented by the condition $|\mu_1 - \mu_2| t_L/\hbar < 1$. In addition to these standard criteria, the potential difference $\delta\mu$ between the superconductors must be small enough that inelastic Andreev scattering from superconductors at different potentials can be ignored and that the change in ϕ during a time t_L is much less than 2π. The latter condition ensures that transport coefficients depend only parametrically on ϕ and yields $\delta\mu\ t_L/\hbar < 1$. The former arises because, if μ is identified with the chemical potential of one superconductor, Andreev scattering from the other involves a quasi-particle energy change of magnitude $\delta\mu$. In a time t_L, a quasi-particle undergoes of the order of t_L/t_a such events, where for superconductors separated by a distance a, $t_a = a^2/D$ and therefore a spread in quasi-particle energy of the order of $\delta E = \delta\mu\ (t_L/t_a)^2$ is produced. If this is not to de-phase the quasi-particles, the condition $\delta E\ t_L/\hbar < 1$ must be satisfied. For $L/a > 1$, this is more stringent than the condition for parametric dependence on ϕ and in the linear response regime is satisfied when $\delta\mu\ (L/a)^2 < |\mu_1 - \mu_2|$. This can occur when a line joining the superconducting inclusions is approximately at 90° to a line joining the external probes, as shown in figure 1, and therefore such devices might be termed 'transverse quantum interference devices', or perhaps 'inverse SQUIDs'.

2. Golden rules for Andreev scattering

For a highly disordered sample, to evaluate the right-hand side of equation (1.3), consider first the normal disordered solid, obtained by setting the superconducting order parameter $\Delta(\boldsymbol{r})$ to zero. In this limit, if the scattering region extends from $x = 0$ to $x = L$, then for $x < 0$ the state $\psi_{j,E}(\boldsymbol{r})$ corresponding to a unit incident flux of particles of energy E, from left to right in channel j, is of the form

$$\psi_{j,E}(\boldsymbol{r}) = \sum_{l=1}^{N} [A_{lj} \exp(\mathrm{i}k_l x) + B_{lj} \exp(-\mathrm{i}k_l x)] \hat{\psi}_l(y, z). \tag{2.1}$$

In this expression, if $v_{j,\mathrm{p}}(E)$ ($v_{j,\mathrm{h}}(E)$) is the group velocity for particles (holes) of energy E in channel j, an incoming plane wave of unit flux along channel j is obtained by choosing $A_{l,j} = \delta_{l,j}(v_{j,\mathrm{p}}(E))^{-1/2}$ and for the case of rectangular leads of cross section $M^2 = d_1 d_2$,

$$\hat{\psi}_l(y,z) = (2/M)\sin(n_l \pi y/d_1)\sin(n_l' \pi z/d_2).$$

If $\psi_{j,E}(\boldsymbol{r})$ is represented by a $2N$-component column vector $|\psi_j\rangle$ formed from the coefficients $\{A_{lj}, B_{lj}\}$, then for $x > L$, $\psi_{j,E}(\boldsymbol{r})$ corresponds to the vector $|\psi_j'\rangle = \hat{\mathsf{T}}|\psi_j\rangle$, where $\hat{\mathsf{T}}$ is the transfer matrix of the normal system [11]. For $0 < x < L$, $\psi_{j,E}(\boldsymbol{r})$ is obtained by solving the Schrödinger equation subject to the boundary condition (2.1).

To lowest order in $\Delta(\boldsymbol{r})$, for an incident particle of energy E incident from the left, the coefficients R_a, T_a can now be evaluated from the following Golden Rules for Andreev scattering,

$$(R_a)_{i,j} = \left|\hbar^{-1}\int \mathrm{d}^3r\ \psi^*_{i,-E}(\boldsymbol{r})\Delta^*(\boldsymbol{r})\psi_{j,E}(\boldsymbol{r})\right|^2 \tag{2.2}$$

$$(T_a)_{i,j} = \left|\hbar^{-1}\int \mathrm{d}^3r\ \phi^*_{i,-E}(\boldsymbol{r})\Delta^*(\boldsymbol{r})\psi_{j,E}(\boldsymbol{r})\right|^2 \tag{2.3}$$

where all spatial integrals extend only over the scattering region $0 \leqslant x \leqslant L$ and in the last expression $\phi_{i,-E}(\boldsymbol{r})$ is the state corresponding to a unit incident particle flux of energy $-E$ from the right. Expressions for R_a' and T_a' are obtained by interchanging ϕ and ψ in these equations. These expressions are useful, because they allow one to extract information about transport properties of the system with inclusions, from a knowledge of the scattering properties of the normal embedding material.

Before evaluating these expressions, a derivation of exact results for transmission and reflection coefficients will be given; these reduce to equations (2.2) and (2.3) in lowest order. To this end it is convenient to write the Bogoliubov–de Gennes equation in the form

$$(\bar{\mathsf{H}}_0 + \bar{\mathsf{H}}_1)\begin{pmatrix}\bar{\psi}_{i,E}(\boldsymbol{r})\\ \bar{\phi}_{i,E}(\boldsymbol{r})\end{pmatrix} = E\begin{pmatrix}\bar{\psi}_{i,E}(\boldsymbol{r})\\ \bar{\phi}_{i,E}(\boldsymbol{r})\end{pmatrix} \tag{2.4}$$

where $\bar{\mathsf{H}}_0$ describes the normal disordered system in the absence of superconductivity and $\bar{\mathsf{H}}_1$ describes scattering due to the superconducting order parameter:

$$\bar{\mathsf{H}}_0(\boldsymbol{r}) = \begin{pmatrix} H_0(\boldsymbol{r}) & 0\\ 0 & -H_0(\boldsymbol{r})\end{pmatrix} \tag{2.5a}$$

and

$$\bar{\mathsf{H}}_1(\boldsymbol{r}) = \begin{pmatrix} 0 & \Delta(\boldsymbol{r})\\ \Delta^*(\boldsymbol{r}) & 0\end{pmatrix}. \tag{2.5b}$$

Starting from the eigenstates

$$\begin{pmatrix}\psi_{j,E}(\boldsymbol{r})\\ 0\end{pmatrix} \qquad \begin{pmatrix}0\\ \phi_{j,E}(\boldsymbol{r})\end{pmatrix}$$

of the normal disordered system described by $\bar{\mathsf{H}}_0$, the solution to equation (2.4), corresponding to an incoming particle from the right, along channel j, is

$$\begin{pmatrix} \bar{\psi}_{j,E}(\boldsymbol{r}) \\ \bar{\phi}_{j,E}(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \psi_{j,E}(\boldsymbol{r}) \\ 0 \end{pmatrix} + \hbar^{-1} \int \mathrm{d}^3 r' \, \mathsf{G}^+(\boldsymbol{r}, \boldsymbol{r}', E) \bar{\mathsf{H}}_1(\boldsymbol{r}') \begin{pmatrix} \bar{\psi}_{j,E}(\boldsymbol{r}') \\ \bar{\phi}_{j,E}(\boldsymbol{r}') \end{pmatrix} \tag{2.6}$$

where the diagonal matrix $\mathsf{G}^{\pm}(\boldsymbol{r}, \boldsymbol{r}', E)$ is of the form

$$\mathsf{G}^{\pm}(\boldsymbol{r}, \boldsymbol{r}', E) = \begin{pmatrix} G_{\mathrm{p}}^{\pm}(\boldsymbol{r}, \boldsymbol{r}', E) & 0 \\ 0 & G_{\mathrm{h}}^{\pm}(\boldsymbol{r}, \boldsymbol{r}', E) \end{pmatrix}$$

and satisfies

$$(E\mathsf{I} - \bar{\mathsf{H}}_0(\boldsymbol{r}))\mathsf{G}^{\pm}(\boldsymbol{r}, \boldsymbol{r}', E) = \hbar \, \delta(\boldsymbol{r} - \boldsymbol{r}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

with superscripts + (−) denoting outgoing (incoming) solutions. Iterating equation (2.6) yields the equivalent expression

$$\begin{pmatrix} \bar{\psi}_{j,E}(\boldsymbol{r}) \\ \bar{\phi}_{j,E}(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \psi_{j,E}(\boldsymbol{r}) \\ 0 \end{pmatrix} + \hbar^{-1} \int \mathrm{d}^3 r' \, \mathrm{d}^3 r'' \, \mathsf{G}^+(\boldsymbol{r}, \boldsymbol{r}', E) \mathsf{T}^+(\boldsymbol{r}', \boldsymbol{r}'') \begin{pmatrix} \psi_{j,E}(\boldsymbol{r}'') \\ 0 \end{pmatrix} \tag{2.8}$$

where

$$\mathsf{T}^{\pm}(\boldsymbol{r}', \boldsymbol{r}'') = \bar{\mathsf{H}}_1(\boldsymbol{r}')\delta(\boldsymbol{r}' - \boldsymbol{r}'') + \hbar^{-1} \int \mathrm{d}^3 r''' \, \bar{\mathsf{H}}_1(\boldsymbol{r}') \mathsf{G}^{\pm}(\boldsymbol{r}', \boldsymbol{r}''', E) \mathsf{T}^{\pm}(\boldsymbol{r}''', \boldsymbol{r}'') \tag{2.9}$$

To obtain Andreev scattering coefficients, one notes that for $x < 0$ or $x > L$, where $\bar{\phi}_{j,E}(\boldsymbol{r})$ is a sum of outgoing plane waves, the amplitude of this scattered hole state in channel i is

$$\bar{\phi}_{ij,E}(x) = \int \mathrm{d}y \, \mathrm{d}z \, \hat{\psi}_i^*(y, z) \bar{\phi}_{j,E}(\boldsymbol{r}).$$

Hence

$$(R_a)_{i,j} = v_{i,\mathrm{h}}(E) |\bar{\phi}_{ij,E}(0)|^2 \tag{2.10}$$

and

$$(T_a)_{i,j} = v_{i,\mathrm{h}}(E) |\bar{\phi}_{ij,E}(L)|^2. \tag{2.11}$$

Similarly to obtain normal scattering coefficients, one notes that for $x < 0$ or $x > L$, the amplitude of the scattered particle state in channel i is

$$\bar{\psi}_{ij,E}(x) = \int \mathrm{d}y \, \mathrm{d}z \, \hat{\psi}_i^*(y, z) \bar{\psi}_{j,E}(\boldsymbol{r}).$$

Hence

$$(T_0)_{i,j} = v_{i,\mathrm{p}}(E)|\bar{\psi}_{ij,E}(L)|^2 \tag{2.12}$$

and

$$(R_0)_{i,j} = v_{i,\mathrm{p}}(E)|\bar{\psi}_{ij,E}(0) - \delta_{ij}\psi_j^0(0)|^2 \tag{2.13}$$

where in the last expression, $\psi^0_{j,E}(x) = (v_{j\mathrm{p}}(E))^{-1/2} \exp \mathrm{i}k_j(x)$ is the longitudinal component of the incoming plane wave along channel j and has been subtracted from $\bar{\psi}_{ij,E}(x)$ to yield the outgoing component for $x \leqslant 0$.

Equations (2.10) to (2.13) are exact results and form a convenient starting point for developing diagrammatic expansions for R_a and T_a. They are analogues of a formula by Fisher and Lee [12] for transmission in normal disordered systems.

The lowest order, equations (2.2) and (2.3) are obtained by retaining only the first term on the right-hand side of equation (2.9) and noting that $G_h^+(\boldsymbol{r},\boldsymbol{r}',E) = [G_p^+(\boldsymbol{r}',\boldsymbol{r},-E)]^*$. This yields

$$\bar{\phi}_{ij,E}(x) = \hbar^{-1} \int \mathrm{d}^3 r' \, [G^+_{\mathrm{p},i}(\boldsymbol{r}',x,-E)]^* \Delta^*(\boldsymbol{r}')\psi_{j,E}(\boldsymbol{r}') \tag{2.14}$$

where

$$G^+_{\mathrm{p},i}(\boldsymbol{r}',x,E) = \int \mathrm{d}x \, \mathrm{d}y \, \hat{\psi}_i(y,z) G_p^+(\boldsymbol{r}',\boldsymbol{r},E).$$

Equation (2.2) follows from the fact that apart from an irrelevant phase factor, for $x \leqslant 0$ and $x' > 0$,

$$G^+_{\mathrm{p},i}(\boldsymbol{r}',x,-E) = (v_{i\mathrm{p}}(-E))^{-1/2}\psi_{i,E}(\boldsymbol{r}')$$

and that $v_{i\mathrm{h}}(E) = v_{i\mathrm{p}}(-E)$. Similarly equation (2.3) follows from the fact that apart from a phase factor, for $x \geqslant L$ and $x' < L$,

$$G^+_{\mathrm{p},i}(\boldsymbol{r}',x,-E) = (v_{i\mathrm{p}}(-E))^{-1/2}\phi_{i,-E}(\boldsymbol{r}').$$

Equation (2.2) shows in a transparent manner the origin of periodic dependence of R_a on the phase difference between superconducting inclusions. To see this, consider the simplest case of narrow external leads with only a single zero-energy channel. At zero temperature, where only the $E = 0$ limit is of interest, equation (2.2) yields

$$R_a = (R_a)_{1,1} = \left| \hbar^{-1} \int \mathrm{d}^3 r \, |\psi^*_{1,0}(\boldsymbol{r})|^2 \Delta^*(\boldsymbol{r}) \right|^2. \tag{2.15}$$

If $\Delta(\boldsymbol{r}) = \Delta \exp(\mathrm{i}\phi_1)$ for the region Ω_1 occupied by superconductor 1 and $\Delta(\boldsymbol{r}) = \Delta \exp(\mathrm{i}\phi_2)$ for the region Ω_2 occupied by superconductor 2, then one immediately obtains

$$R_a = |A_1 \exp(\mathrm{i}\phi_1) + A_2 \exp(\mathrm{i}\phi_2)|^2 \tag{2.16}$$

where

$$A_i = \Delta\hbar^{-1}\int_{\Omega_i} \mathrm{d}^3r\, |\phi^*_{1,0}(r)|^2. \tag{2.17}$$

A similar result is obtained for R'_a and for the case where $R_a \ll R'_a$, since A_1 and A_2 are positive, one recovers equation (1.1), with $\theta_{11} = 0$. Whatever the relative values of R_a, R'_a, this demonstrates that G is periodic with period 2π. Equation (2.15) also illustrates that, whereas the values of A_i depend on the particular realization of the disorder, particle–hole symmetry at $E = 0$, which leads to a positive integrand in equation (2.16), produces a phase $\theta_{11} = 0$, which is independent of microscopic disorder. Hence the ensemble averaged conductance is periodic with period 2π, in contrast with the weak localization result of [4,5]. Thus a periodicity of π is not a general property of the ensemble averaged conductance.

3. Evaluation of Andreev scattering coefficients in the presence of many channels

For leads with many channels, to obtain an expression for

$$R_a = \sum_{i,j}(R_a)_{i,j}$$

in the highly disordered limit, a transformation to eigenstates $\{|f_j\rangle\}$ of $\hat{\mathrm{T}}^\dagger\hat{\mathrm{T}}$ is appropriate [11]. The corresponding eigenvalues are of the form $\lambda_m = \exp(-\alpha_m L)$, where for $1 \leqslant m \leqslant N$, the exponents $\{\alpha_m\}$ are positive and for $N+1 \leqslant m \leqslant 2N$, $\alpha_m = -\alpha_{m-N}$. If the associated real space functions $f_j(r)$ are written $f_j(r) = g_j(r)\exp(-\alpha_j x)$ then the non-unitary transformation takes the form

$$\psi_j(r) = \sum_{m=1}^{2N} u_{m,j}\exp(-\alpha_m x) g_m(r) \tag{3.1}$$

where $u_{m,j} = \langle f_m|\psi_j\rangle$. Since only the limit $E = 0$ is of interest here, the subscript E has been omitted. The fact that eigenchannels of $\hat{\mathrm{T}}^\dagger\hat{\mathrm{T}}$ are associated with exponentially decaying eigenvalues leads to great simplifications in the strong disorder limit. For example, when $x > L$, $|\psi(r)|^2 \sim T_0 \sim \exp(-2\alpha_1 L)$, where α_1 is the smallest positive exponent. Therefore the coefficients $u_{m,j}$ for $m > N$ are of order $\exp(-\alpha_1 - |\alpha_m|)$ and can be ignored. The transformed expression for R_a then becomes

$$R_a = \sum_{\substack{m,n=1\\ m',n'=1}}^{N} \hat{\delta}_{m,m'}\hat{\delta}_{n,n'} I_{m',n} I^*_{m,n'} \tag{3.2}$$

where

$$\hat{\delta}_{m,m'} = \sum_{i=1}^{N} u_{m,i} u^*_{m',i}$$

In the case of N_s spatially separate superconductors, where in the region occupied by superconductor s the order parameter has the form $\Delta(\boldsymbol{r}) = \Delta \exp \mathrm{i}\phi_s$, the integrals $I_{m,n}$ take the form of a sum of interfering contributions from each superconducting inclusion

$$I_{m,n} = \sum_{s=1}^{N_s} I_{m,n}^{(s)}. \tag{3.3}$$

If as in figure 1(*a*), superconductor s occupies a region of space of volume $\Omega^{(s)}$ between $x = L_s$ and $x = L_s + l_s$, then

$$I_{m,n}^{(s)} = (\hbar v_{\mathrm{F}} M^2)^{-1} \Delta \overline{\Omega}_{m,n}^{(s)} \exp[-(\alpha_m + \alpha_n) L_s + \mathrm{i}(\phi_s + \theta_{m,n}^{(s)})]. \tag{3.4}$$

where v_{F} is the Fermi velocity and

$$\overline{\Omega}_{m,n}^{(s)} = \Omega^{(s)} \chi_{m,n}^{(s)} \{1 - \exp[-(\alpha_m + \alpha_n) l_s\} / \{(\alpha_m + \alpha_n) l_s\}. \tag{3.5}$$

In this expression the value of both the real, dimensionless number $\chi_{m,n}^{(s)}$ and the phase $\theta_{m,n}^{(s)}$ depend on the detailed form of the eigenstates $g_n(\boldsymbol{r})$, $g_m(\boldsymbol{r})$ and on the precise shape of the superconductor s.

Equations (3.2) to (3.4) clearly demonstrate that interference from $N_s > 1$ superconductors leads to an oscillatory contribution to quasi-particle transport coefficients. Again in the high disorder limit, where the spacing $(\alpha_2 - \alpha_1)$ is greater than $1/L$, gross simplifications arise, because the right-hand side of equation (3.2) is dominated by the smallest term and for $N_s = 2$ reduces to

$$R_a = A_1^2 + A_2^2 + 2A_1A_2 \cos(\phi + \theta_{11}) \tag{3.6}$$

where $A_s = |I_{1,1}^{(s)}|$ and $\theta_{mn} = \theta_{m,n}^{(1)} - \theta_{m,n}^{(2)}$. For weaker disorder, where the number N_λ of eigenchannels contributing significantly to R_a is greater than unity, the phase average of R_a grows linearly with N_λ, whereas if the random phases θ_{mn} are uncorrelated, the prefactor of the oscillatory term will increase as $N_\lambda^{1/2}$. Therefore the effect is present even in the many-channel case, although the relative amplitude of oscillation is diminished. It is interesting to note that in the strong disorder limit, the probability distribution of the amplitudes A_s will possess a long tail, reflecting the log–normal distribution of $\exp\{\alpha L_s\}$ [12]. Hence the relative size of the oscillatory term will exhibit large fluctuations amongst members of the same statistical ensemble and for a given sample the sum $1/R_a + 1/R'_a$ will be dominated by the smallest of R_a, R'_a. It should be noted that since the expression (2.3) for T_a involves a product of states which decay from the left and right respectively,

$$T_a \sim T_0 \sim \exp[-2\alpha_1(L - 2L_s)] \sim \exp[-2\alpha_1 l_s].$$

Consequently for inclusions of size $l_s > \alpha_1$, the reduced formula (2.2) can be employed, to yield $G \simeq 2(R_a)_{\min}$, where $(R_a)_{\min}$ is the smaller of R_a, R'_a.

4. Discussion

When the phases are decoupled, the oscillatory behaviour in equation (3.6) bears a resemblance to the AC Josephson effect, but differs from it in several respects. For example as noted above, the periodicity here is in the conductance G and therefore the amplitude as well as the frequency of current oscillations will increase with applied voltage. In addition, as is evident from equations (3.3) and (3.4), the effect is not restricted to two superconducting inclusions, and therefore for more than two inclusions, to facilitate observation of this phenomenon, it should be possible to produce low-frequency beats between higher-frequency oscillations associated with different pairs of superconductors. Furthermore, there may be one or two orders of magnitude difference between the superconducting transition temperature at which Josephson effects can occur and the temperature at which mesoscopic effects become observable. If a small coupling exists between two superconducting inclusions, then as the external potential difference $\mu_1 - \mu_2$ is increased, the above time-dependent behaviour will be preceded by a time-independent regime. For small $\mu_1 - \mu_2$, the inclusions will remain coherently coupled, with a time-independent phase difference ϕ associated with a small, sub-critical supercurrent flowing from one inclusion to the other. In this regime, depending on the value of the random phase θ_{11} in equation (10), G may vary non-monotonically with $\mu_1 - \mu_2$ and as ϕ increases from 0 to π may therefore exhibit negative differential resistance.

Acknowledgments

This work is supported by the SERC and the EC SCIENCE program. I wish to thank the CMP group at Michigan State University for their kind hospitality, while this work was carried out. This work has benefited from conversations with W P Pratt Jr, P D Beale, J C Price, V C Hui and S J Robinson.

References

[1] Beenakker C W J and van Houten H 1991 *Phys. Rev. Lett.* **66** 3056
[2] Beenakker C W J 1991 *Phys. Rev. Lett.* **67** 3836
[3] Farusaki A, Takayanagi H and Tsukada M *Phys. Rev. Lett.* **67** 132
[4] Al'tshuler B L and Spivak B Z 1987 *Sov. Phys.–JETP* **65** 343
[5] Spivak B Z and Khmel'nitskii D E 1982 *JETP Lett* **35** 413
[6] Landauer R 1970 *Phil. Mag.* **21** 863
[7] Buttiker M, Imry Y, Landauer R and Pinhas S 1985 *Phys. Rev.* B **31** 6207
[8] Buttiker M 1988 *IBM J. Res. Dev.* **32** 317
[9] Lambert C J 1991 *J. Phys.: Condens. Matter* **3** 6579
[10] Andreev A F 1964 *Sov. Phys.–JETP* **19** 1228
[11] Stone A D, Mello P A, Muttalib K A and Pichard J-L 1991 *Mesoscopic Phenomena In Solids* ed B L Al'tschuler, P A Lee and R A Webb (Amsterdam: North Holland)
[12] Fisher D S and Lee P A 1981 *Phys. Rev.* B **23** 6851